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1995 J. Phys. A: Math. Gen. 28 2759

(http://iopscience.iop.org/0305-4470/28/10/009)

J. Phys. A: Math. Gen. 28 (1995) 2759–2770. Printed in the UK

Exact solution and interfacial tension of the six-vertex model with anti-periodic boundary conditions

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Received 20 March 1995

Abstract. We consider the six-vertex model with anti-periodic boundary conditions across a finite strip. The row-to-row transfer matrix is diagonalized by the 'commuting transfer matrices' method. From the exact solution we obtain an independent derivation of the interfacial tension of the six-vertex model in the anti-ferroelectric phase. The nature of the corresponding integrable boundary condition on the XXZ spin chain is also discussed.

1. Introduction and main results

The six-vertex model and related spin-$\frac{1}{2}$ XXZ chain play a central role in the theory of exactly solved lattice models [1]. Typically the six-vertex model is 'solved' by diagonalizing the row-to-row transfer matrix with periodic boundary conditions. Several methods have evolved for doing this, including the coordinate Bethe ansatz [1, 2], the algebraic Bethe ansatz [3, 4], and the analytic ansatz [5]. All of these methods rely heavily on the conservation of arrow flux from row to row of the lattice.

In terms of the vertex weights (see figure 1)

$$a = \rho\,\sinh\tfrac{1}{2}(\lambda - v) \qquad b = \rho\,\sinh\tfrac{1}{2}(\lambda + v) \qquad c = \rho\,\sinh\lambda \tag{1.1}$$

the transfer matrix eigenvalues on a strip of width N are given by [1]

$$\Lambda(v)q(v) = \phi(\lambda - v)q(v + 2\lambda') + \phi(\lambda + v)q(v - 2\lambda') \tag{1.2}$$

where

$$\lambda' = \lambda - \mathrm{i}\pi \tag{1.3}$$

$$\phi(v) = \rho^N \sinh^N\left(\tfrac{1}{2}v\right) \tag{1.4}$$

$$q(v) = \prod_{k=1}^{n} \sinh\tfrac{1}{2}(v - v_k)\,. \tag{1.5}$$

The Bethe ansatz equations follow from (1.2) as

$$\frac{\phi(\lambda - v_j)}{\phi(\lambda + v_j)} = -\frac{q(v_j - 2\lambda')}{q(v_j + 2\lambda')} \qquad j = 1, \ldots, n\,. \tag{1.6}$$

The integer n labels the sectors of the transfer matrix.

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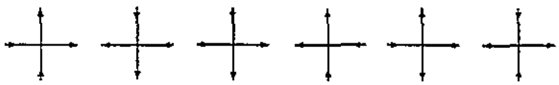


Figure 1. Standard vertex configurations and corresponding weights.

Here we consider the same six-vertex model with *anti-periodic* boundary conditions. That such boundary conditions should preserve integrability is known through the existence of commuting transfer matrices [6]. However, the solution itself has not been found previously. In section 2 we solve the anti-periodic six-vertex model by the 'commuting transfer matrices' method [1]. This approach has its origin in the solution of the more general eight-vertex model [7], which, like the present problem, no longer enjoys arrow conservation. We find the transfer matrix eigenvalues to be given by

$$\Lambda(v)q(v)=\phi(\lambda-v)q(v+2\lambda')-\phi(\lambda+v)q(v-2\lambda') \tag{1.7}$$

where now

$$q(v)=\prod_{k=1}^{N}\sinh\tfrac{1}{4}(v-v_k)\,. \tag{1.8}$$

In this case the Bethe ansatz equations are

$$\frac{\phi(\lambda-v_j)}{\phi(\lambda+v_j)}=\frac{q(v_j-2\lambda')}{q(v_j+2\lambda')}\qquad j=1,\ldots,N\,. \tag{1.9}$$

In contrast with the periodic case the number of roots is fixed at N.

In section 3 we use this solution to derive the interfacial tension s of the six-vertex model in the anti-ferroelectric regime. Defining $x=\mathrm{e}^{-\lambda}$, our final result is

$$\mathrm{e}^{-s/k_BT}=2x^{1/2}\prod_{m=1}^{\infty}\left(\frac{1+x^{4m}}{1+x^{4m-2}}\right)^2 \tag{1.10}$$

in agreement with the result obtained from the asymptotic degeneracy of the two largest eigenvalues [1, 8].

With anti-periodic boundary conditions on the vertex model, the related XXZ Hamiltonian is

$$\mathcal{H}=\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\cosh\lambda\;\sigma_j^z\sigma_{j+1}^z\right) \tag{1.11}$$

where σ^x, σ^y and σ^z are the usual Pauli matrices, with boundary conditions

$$\sigma_{N+1}^x=\sigma_1^x\qquad\sigma_{N+1}^y=-\sigma_1^y\qquad\sigma_{N+1}^z=-\sigma_1^z\,. \tag{1.12}$$

This boundary condition has appeared previously, being of relevance to the operator content of the Ashkin–Teller chain [9], and is amongst the class of toroidal boundary conditions for which the operator content of the XXZ chain has been determined by finite-size studies [9, 10]. Thus we see it is an integrable boundary condition, with the eigenvalues of the Hamiltonian following from (1.7) in the usual way [1], with result

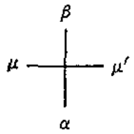
$$E=N\cosh\lambda-\sum_{j=1}^{N}\frac{2\cosh\frac{1}{2}\lambda\,\sinh\lambda}{\sinh\frac{1}{2}v_j+\sinh\frac{1}{2}\lambda}\,. \tag{1.13}$$

We anticipate that the approach adopted here may also be successful in solving other models without arrow conservation. The solution given here can be extended, for example, to the spin-S generalization of the six-vertex model/XXZ chain [11].

2. Exact solution

To obtain the result (1.7) we adapt, where appropriate, the derivation of the periodic result (1.2) (specifically, we refer the reader to chapter 9 of [1]).

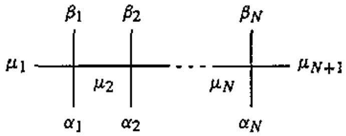
We depict a vertex and its corresponding Boltzmann weight graphically, as

$$w(\mu, \alpha|\beta, \mu') =$$


where the bond 'spins' μ, α, β and μ' are each $+1$ if the corresponding arrow points up or to the right and -1 if the arrow points down or to the left. Thus in terms of the parametrization (1.1) the non-zero vertex weights are

$$\begin{aligned} w(+,+|+,+) &= w(-,-|,-,-) = a \\ w(+,-|-,+) &= w(-,+|,+,-) = b \\ w(+,-|+,-) &= w(-,+|,-,+) = c\,. \end{aligned} \tag{2.1}$$

The row-to-row transfer matrix T has elements

$$T_{\alpha\beta} = \sum_{\mu_1} \cdots \sum_{\mu_N} \tag{2.2}$$


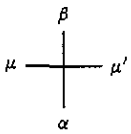
where $\alpha = \{\alpha_1, \ldots, \alpha_N\}$, $\beta = \{\beta_1, \ldots, \beta_N\}$, and the anti-periodic boundary condition is such that $\mu_{N+1} = -\mu_1$. Now consider an eigenvector y of the form

$$y = g_1 \otimes g_2 \otimes \cdots \otimes g_N \tag{2.3}$$

where $g_i(\alpha_i)$ are two-dimensional vectors. From equation (2.2) the product Ty can be written as

$$(Ty)_\alpha = \mathrm{Tr}\,[G_1(\alpha_1)G_2(\alpha_2) \cdots G_N(\alpha_N)S] \tag{2.4}$$

where $G_i(\pm)$ are 2×2 matrices with elements

$$G_i(\alpha)_{\mu\mu'} = \sum_\beta \quad [\text{vertex}] \quad g_i(\beta)\,. \tag{2.5}$$


The appearance of the spin reversal operator

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.6}$$

in equation (2.4) is the key difference with the periodic case. However, it does not effect much of the working. Using equations (2.1) and (2.5) we have

$$G_i(+) = \begin{pmatrix} a\,g_i(+) & 0 \\ c\,g_i(-) & b\,g_i(+) \end{pmatrix} \qquad G_i(-) = \begin{pmatrix} b\,g_i(-) & c\,g_i(+) \\ 0 & a\,g_i(-) \end{pmatrix}. \tag{2.7}$$

In particular, there still exist the same 2×2 matrices $P_1, \ldots, P_N$ such that

$$G_i(\alpha) = P_i H_i(\alpha) P_{i+1}^{-1} \tag{2.8}$$

where P_i and H_i are of the form

$$P_i = \begin{pmatrix} p_i(+) & \star \\ p_i(-) & \star \end{pmatrix} \qquad H_i(\alpha) = \begin{pmatrix} g_i'(\alpha) & g_i'''(\alpha) \\ 0 & g_i''(\alpha) \end{pmatrix}. \tag{2.9}$$

As for the periodic case, (2.8) follows from the local 'pair-propagation through a vertex' property, i.e. the existence of $g_i(\alpha)$, $g_i'(\alpha)$, $p_i(\alpha)$, $p_{i+1}(\alpha)$ such that

$$\sum_{\beta,\mu'} w(\mu,\alpha|\beta,\mu')g_i(\beta)p_{i+1}(\mu') = g_i'(\alpha)p_i(\mu) \tag{2.10}$$

for $\alpha, \mu = \pm 1$. The available parameters are [1]

$$\begin{aligned} g_i(+) &= 1 & g_i(-) &= r_i\,\mathrm{e}^{(\lambda+v)\sigma_i/2} \\ g_i'(+) &= a & g_i'(-) &= -a\,r_i\,\mathrm{e}^{(3\lambda+v)\sigma_i/2} \\ p_i(+) &= 1 & p_i(-) &= r_i \end{aligned} \tag{2.11}$$

where $\sigma_i = \pm 1$ and

$$r_i = (-)^i r\,\mathrm{e}^{\lambda(\sigma_1+\cdots+\sigma_{i-1})}. \tag{2.12}$$

However, p_{N+1} needs to be different from the periodic case (where $p_{N+1} = p_1$). The anti-periodicity suggests that we require

$$\begin{pmatrix} p_{N+1}(+) \\ p_{N+1}(-) \end{pmatrix} = h \begin{pmatrix} p_1(-) \\ p_1(+) \end{pmatrix} \tag{2.13}$$

where h is some scalar. Since we already require $p_i(+) = 1$ and $p_i(-) = r_i$, we must have

$$r_1 = \frac{1}{h} = -r \qquad \text{and} \qquad r_{N+1} = h = -\frac{1}{r}. \tag{2.14}$$

In addition,

$$r^2 = (-)^N \mathrm{e}^{-\lambda(\sigma_1+\cdots+\sigma_N)}. \tag{2.15}$$

To proceed further, we write P_1 and P_{N+1} in full,

$$P_1 = \begin{pmatrix} p_1(+) & q_1(+) \\ p_1(-) & q_1(-) \end{pmatrix} \qquad P_{N+1} = \begin{pmatrix} h\,p_1(-) & q_{N+1}(+) \\ h\,p_1(+) & q_{N+1}(-) \end{pmatrix}. \tag{2.16}$$

Then

$$\begin{aligned} P_{N+1}^{-1} S P_1 &= \frac{1}{\det P_{N+1}} \begin{pmatrix} q_{N+1}(-) & -q_{N+1}(+) \\ -h\,p_1(+) & h\,p_1(-) \end{pmatrix} \\ &= \begin{pmatrix} 1/h & \star \\ 0 & -h\frac{\det P_1}{\det P_{N+1}} \end{pmatrix}. \end{aligned} \tag{2.17}$$

Putting the pieces together we then have

$$\begin{aligned} (Ty)_\alpha &= \mathrm{Tr}\left[P_1 H_1(\alpha_1)\ldots H_N(\alpha_N) P_{N+1}^{-1} S\right] \\ &= \frac{1}{h} g_1'(\alpha_1)\ldots g_N'(\alpha_N) - h\frac{\det P_1}{\det P_{N+1}} g_1''(\alpha_1)\ldots g_N''(\alpha_N). \end{aligned} \tag{2.18}$$

However, as for the periodic case, we have

$$g_i''(\alpha_i) = ab\frac{g_i^2(\alpha_i)\det P_{i+1}}{g_i'(\alpha_i)\det P_i} \tag{2.19}$$

which follows from (2.7)–(2.9). Thus

$$(Ty)_\alpha = -r g_1'(\alpha_1)\ldots g_N'(\alpha_N) + \frac{1}{r}(ab)^N \frac{g_1^2(\alpha_1)\ldots g_N^2(\alpha_N)}{g_1'(\alpha_1)\ldots g_N'(\alpha_N)}. \tag{2.20}$$

At this point it is more convenient to write

$$y(v) = h_1(v) \otimes h_2(v) \otimes \cdots \otimes h_N(v) \tag{2.21}$$

where we have defined

$$h_i(v) = \begin{pmatrix} 1 \\ r_i\, e^{\frac{1}{2}(\lambda+v)\sigma_i} \end{pmatrix}. \tag{2.22}$$

The result (2.20) can then be more conveniently written as

$$T(v)y(v) = -ra^N y(v+2\lambda') + \frac{1}{r} b^N y(v-2\lambda'). \tag{2.23}$$

Also let

$$y^{\pm}_{\sigma}(v) = \exp\left(-\frac{v}{4}\sum_{i=1}^{N}\sigma_i\right) y(\alpha) \tag{2.24}$$

with $r = \mp \exp(\frac{\lambda'}{2}\sum_{i=1}^{N}\sigma_i)$. Then from (2.23) we have

$$T(v)y^{\pm}_{\sigma}(v) = \pm\phi(\lambda - v)y^{\pm}_{\sigma}(v+2\lambda') \mp \phi(\lambda+v)y^{\pm}_{\sigma}(v-2\lambda'). \tag{2.25}$$

To proceed further, let $Q^{\pm}_R(v)$ be a matrix whose columns are a linear combination of $y^{\pm}_{\sigma}$ with different choices of σ (2^N altogether). It follows from (2.25) that

$$T(v)Q^{\pm}_R(v) = \pm\phi(\lambda - v)Q^{\pm}_R(v+2\lambda') \mp \phi(\lambda+v)Q^{\pm}_R(v-2\lambda'). \tag{2.26}$$

One can show that the transpose of the transfer matrix has the property $T(-v) = {}^tT(v)$. With $Q^{\mp}_L(v) = {}^tQ^{\pm}_R(-v)$ it follows from (2.26) that

$$Q^{\pm}_L(v)T(v) = \pm\phi(\lambda - v)Q^{\pm}_L(v+2\lambda') \mp \phi(\lambda+v)Q^{\pm}_L(v-2\lambda'). \tag{2.27}$$

Now let $Q_R(v) = Q^{+}_R(v)$ and $Q_L(v) = Q^{+}_L(v)$†. Then we can show that the 'commutation relations'

$$Q_L(u)Q_R(v) = Q_L(v)Q_R(u) \tag{2.28}$$

hold for arbitrary u and v. This result follows if we can prove that $F_{\sigma\sigma'} = {}^ty^{-}_{\sigma}(-u)y^{+}_{\sigma'}(v)$ is a symmetric function of (u, v) for all choices of σ and σ'. Using equations (2.24), (2.21) and (2.22) this function reads

$$F_{\sigma\sigma'} = \exp\left(\frac{u}{4}\sum_{i=1}^{N}\sigma_i - \frac{v}{4}\sum_{i=1}^{N}\sigma'_i\right)\prod_{j=1}^{N}\Bigg[1 - (-)^{\frac{1}{2}\sum_{i=1}^{N}(\sigma_i+\sigma'_i)} \times \exp\left\{\tfrac{1}{2}(\lambda-u)\sigma_j + \tfrac{1}{2}(\lambda+v)\sigma'_j - \tfrac{1}{2}\lambda\left[\sum_{i=j}^{N}(\sigma_i+\sigma'_i) - \sum_{i=1}^{j-1}(\sigma_i+\sigma'_i)\right]\right\}\Bigg]. \tag{2.29}$$

Now suppose that in σ and σ' there are p pairs $(\sigma_{i_k}, \sigma'_{i_k})$, where $\sigma_{i_k} + \sigma'_{i_k} = 0$ with $k = 1, \ldots, p$. The terms in $F_{\sigma\sigma'}$ which involve these σ_{i_k} (in the prefactor and in the $j = i_k$ terms) are manifestly symmetric in (u, v). The remaining terms are exactly of the form (2.29) with $N \to N - p$ after relabelling of sites. We can thus restrict ourselves to the case where $\sigma_i = \sigma'_i$, $i = 1, \ldots, N'$, for all N'. To prove this case we proceed inductively. From equation (2.28) we have

$$F_{\sigma\sigma} = \prod_{j=1}^{N}\left[e^{\frac{1}{4}(u-v)\sigma_j} - (-)^N e^{\frac{1}{4}(v-u)\sigma_j}\, e^{-\lambda(\sigma_N+\cdots+\sigma_{j+1})}\, e^{\lambda(\sigma_{j-1}+\cdots+\sigma_1)}\right]. \tag{2.30}$$

† Equivalent results are obtained using the other choice of sign.

Let us now denote $F_{\sigma\sigma} = F_N(\sigma_1, \dots, \sigma_N)$. By inspection, $F_1(\sigma_1)$ and $F_2(\sigma_1, \sigma_2)$ are symmetric in (u, v). Suppose $F_{N-2}(\sigma_1, \dots, \sigma_{N-2})$ is symmetric in (u, v) and, furthermore, that $\sigma_k + \sigma_{k+1} = 0$ for some k. Then from (2.30) we have $F_N(\sigma_1, \dots, \sigma_k, -\sigma_k, \dots, \sigma_N) = F_{N-2}(\sigma_1, \dots, \hat{\sigma}_k, \hat{\sigma}_{k+1}, \dots, \sigma_{N-2})$ times a symmetric function of (u, v), which is therefore symmetric in (u, v). This is true for all $1 \leqslant k \leqslant N-1$. The only case left to consider is therefore $\sigma_1 = \sigma_2 = \cdots = \sigma_N$. But from (2.30) we have $F_N(\sigma_1, \sigma_2, \dots, \sigma_{N-1}, \sigma_1) = F_{N-2}(\sigma_2, \dots, \sigma_{N-1})$ times a symmetric function of (u, v), which is again symmetric. Thus by induction on N, the assertion (2.28) follows.

As in the periodic case, we assume that $Q_R(v)$ is invertible at some point $v = v_0$ and define

$$Q(v) = Q_R(v) Q_R^{-1}(v_0) = Q_L^{-1}(v_0) Q_L(v) . \tag{2.31}$$

Then from (2.27) and (2.28) we obtain

$$T(v)Q(v) = Q(v)T(v) = \phi(\lambda - v)Q(v + 2\lambda') - \phi(\lambda + v)Q(v - 2\lambda') \tag{2.32}$$

and $Q(u)Q(v) = Q(v)Q(u)$. This allows $T(v)$, $Q(v)$ and $Q(v \pm 2\lambda')$ to be simultaneously diagonalized, yielding the relation (1.7) for their eigenvalues. The precise functional form of the eigenvalue $q(v)$ of $Q(v)$, given in (1.8), follows from (2.32) by noting that $T(v + 2\pi \mathrm{i}) = -T(v)$ and considering the limits $v \to \pm\infty$.

3. Interfacial tension

In this section we derive the interfacial tension by solving the functional relation (1.7) and integrating over the band of largest eigenvalues of the transfer matrix [12]. We consider the case where N, the number of columns in the lattice, is even. The partition function of the model is expressed in terms of the eigenvalues $\Lambda(v)$ of the row-to-row transfer matrix $T(v)$ as

$$Z = \sum [\Lambda(v)]^M \tag{3.1}$$

where the sum is over all 2^N eigenvalues.

The interfacial tension is defined as follows. Consider a single row of the lattice. For a system with periodic boundary conditions, in the $\lambda \to \infty$ limit we see from (1.1) that the vertex weight c is much greater than the weights a and b, so in this limit, the row can be in one of two possible anti-ferroelectrically ordered ground states. These are made up entirely of spins with Boltzmann weight c, and are related to one another by arrow-reversal.

When we impose anti-periodic boundary conditions, this ground-state configuration is no longer consistent with N even. To ensure the anti-periodic boundary condition, vertices with Boltzmann weight c must occur an odd number of times in each row. Thus the lowest-energy configuration for the row in the $\lambda \to \infty$ limit will consist of $N-1$ vertices with weight c, and one vertex of either types a or b. This different vertex can occur anywhere in the row.

As we add rows to form the lattice, the a or b vertex in each row forms a 'seam' running approximately vertically down the lattice; it can jump from left to right but the mean direction is downwards†. A typical lowest-energy configuration is shown in figure 2. The extra free energy due to this seam is called the interfacial tension. This will grow with the height M of the lattice, so we expect that for large N and M the partition function of the lattice will be of the form

$$Z \sim \exp\left[(-NMf - Ms)/k_B T\right] \tag{3.2}$$

† This is the analogue of the anti-ferromagnetic seam in the Ising model [13].

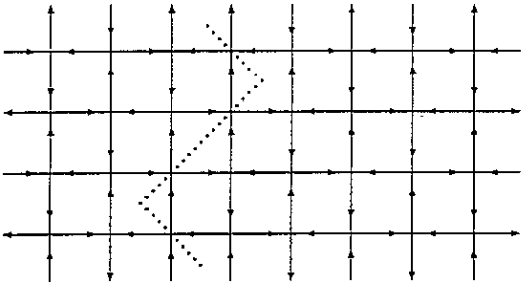


Figure 2. A typical lowest-energy state of the system with N even and anti-periodic boundary conditions. The dotted line indicates the interface dividing the lattice into two domains, each of which is an anti-ferroelectrically ordered ground state.

where f is the normal bulk free energy, and s is the interfacial tension.

We introduce the variables

$$x = \mathrm{e}^{-\lambda} \qquad z = \mathrm{e}^{-v/2}. \tag{3.3}$$

Expressing the Boltzmann weights in terms of z and x, from (1.1) the model is physical when z and x are real, and z lies in the interval

$$x^{1/2} \leqslant z \leqslant x^{-1/2}. \tag{3.4}$$

We consider $\lambda \geqslant 0$ in order that the Boltzmann weights are non-negative, so we must have $x \leqslant 1$. Let

$$\tilde{Q}(z) = \prod_{j=1}^{N}(z - z_j) \tag{3.5}$$

where $z_j = \mathrm{e}^{-v_j/2}$, $j = 1, \ldots, N$, and

$$V(z) = \Lambda(v)(2z\rho^{-1})^N(-)^{N/2}. \tag{3.6}$$

In terms of these variables, the functional relation (1.7) becomes

$$\tilde{Q}(z)V(z) = (1 - z^2x^{-1})^N\tilde{Q}(-zx) - (1 - z^2x)^N\tilde{Q}(-zx^{-1}). \tag{3.7}$$

Both terms on the right-hand side of (3.7) are polynomials in z of degree $3N$, but the coefficients of 1 and z^{3N} vanish, so $z^{-1}V(z)$ is a polynomial in z of degree $2N-2$. We know how to solve equations of this form for both $V(z)$ and $\tilde{Q}(z)$ using Wiener–Hopf factorizations (see [7, 8, 14]). An alternative approach is via root densities [15].

We shall need some idea where the zeros of the polynomials $\tilde{Q}(z)$ and $V(z)$ lie in order to construct the Wiener–Hopf factorizations. From the anti-periodicity of $T(v)$ we see that $V(z)$ is an odd function of z,

$$V(-z) = -V(z) \tag{3.8}$$

so its zeros and poles must occur in plus–minus pairs. To locate the zeros in the z-plane, we consider z to be a free variable, and vary the parameter x, in particular, looking at the limit $x \to 0$.

We find the following; in the $x \to 0$ limit, $N-2$ of the N zeros of $\tilde{Q}(z)$ lie on the unit circle, the other two lying at distances proportional to $x^{1/2}$ and $x^{-1/2}$. For $V(z)$, there is the simple zero at the origin, and two zeros on the unit circle. The remaining $2N-4$ zeros of $V(z)$ are divided into two sets, with $N-2$ of them that approach the origin and $N-2$ that approach ∞ as $x \to 0$. The N zeros of the two polynomials that lie on the unit circle are spaced evenly around the circle.

As x is increased, the zeros of $\tilde{Q}(z)$ and $z^{-1}V(z)$ will all shift. We assume that the distribution of the zeros mentioned above does not change significantly as x increases. Thus

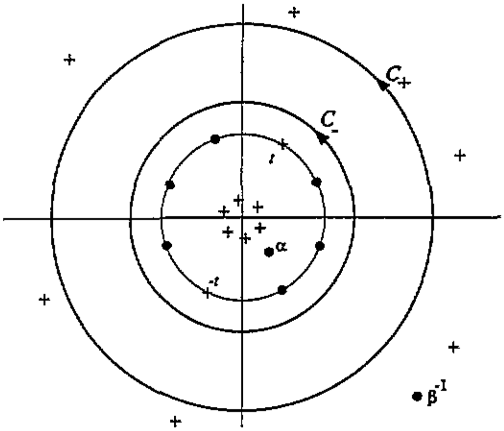


Figure 3. The complex z-plane; the curves C_+ and C_- are indicated, with the unit circle lying inside C_-. The zeros of $\tilde{Q}$ are indicated by (•) and the zeros of $z^{-1}V(z)$ by (+). There are no zeros of either function in the annulus between the contours C_+ and C_-.

the zeros that lie at the origin in the $x \to 0$ limit move out from the origin as x increases, but not so far out as the unit circle, and similarly for the zeros that lie at ∞. Also, the zeros that lie on the unit circle are assumed to stay in some neighbourhood of the unit circle as x increases (we will show that these zeros remain exactly on the unit circle as x increases, which is what happens in the periodic boundary condition case).

Bearing in mind the above comments, we write

$$\tilde{Q}(z) = \tilde{Q}_1(z)(z-\alpha)(z-\beta^{-1}) \tag{3.9}$$

where $\tilde{Q}_1(z)$ is a polynomial of degree $N-2$ whose zeros are O(1) as $x \to 0$, and $\alpha, \beta = \mathrm{O}(x^{1/2})$, so α lies inside the unit circle, β^{-1} outside.

Also, let $V(z) = z(z-t_1)(z-t_2)A(z)B(z)$, where $A(z)$ and $B(z)$ are both polynomials of degree $N-2$, the zeros of $A(z)$ being all the zeros of $V(z)$ that lie inside the unit circle, $B(z)$ containing all those that lie outside, and t_1 and t_2 are the zeros that lie on the unit circle. Since $V(z)$ is an odd function, both $A(z)$ and $B(z)$ must be even functions of z, and we must have $t_1 = -t_2$, so letting $t_1 = -t_2 = t$, we write

$$V(z) = z(z^2-t^2)A(z)B(z)\,. \tag{3.10}$$

Draw the contours C_+ and C_- in the complex z-plane, both oriented in the positive direction, with C_- outside the unit circle, C_+ outside C_-, and such that there are no zeros of $\tilde{Q}(z)$ or $V(z)$ on the boundary of or inside the annulus between C_- and C_+. Then β^{-1} and all the zeros of $B(z)$ lie outside C_+ (see figure 3).

Define $r(z)$ as the quotient of the two terms in the RHS of the functional relation (3.7);

$$r(z) = -\frac{\tilde{Q}(-zx^{-1})(1-z^2x)^N}{\tilde{Q}(-zx)(1-z^2x^{-1})^N} \tag{3.11}$$

($r(z)$ has no zeros or poles on or between the curves C_+ and C_-). Then in the $x \to 0$ limit, we see that $|r(z)| \sim 1/z^N$, so when $|z| > 1$, $|r(z)| < 1$. Thus $\ln[1+r(z)]$ can be chosen to be single-valued and analytic when z lies in the annulus between C_- and C_+. We can therefore make a Wiener–Hopf factorization of $1+r(w)$ by defining the functions $P_+(z)$ and $P_-(z)$ as

$$\ln P_\pm(z) = \pm\frac{1}{2\pi\mathrm{i}}\oint_{C_\pm} \ln\big[1+r(z')\big]\frac{\mathrm{d}z'}{z'-z}\,. \tag{3.12}$$

Then $P_+(z)$ is an analytic and non-zero (ANZ) function of z for z inside C_+, and $P_-(z)$ is an ANZ function of z for z outside C_-. As $|z| \to \infty$, we note that $P_-(z) \to 1$. When z is inside the annulus between C_- and C_+, we have, by Cauchy's integral formula

$$1 + r(z) = P_+(z)\, P_-(z) = \frac{V(z)\tilde{Q}(z)}{\tilde{Q}(-zx)(1 - z^2x^{-1})^N}. \tag{3.13}$$

We then define the functions $V_\pm(z)$;

$$V_+(z) = P_+(z)\tilde{Q}(-zx)/(z - \beta^{-1}) \tag{3.14}$$

$$V_-(z) = P_-(z)(1 - z^2x^{-1})^N/[\tilde{Q}_1(z)(z - \alpha)] \tag{3.15}$$

where $V_+(z)$ is an ANZ function of z for z inside C_+, $V_-(z)$ an ANZ function of z for z outside C_-. We have split $V(z)$ into two factors, $V_+(z)$ and $V_-(z)$, with $V(z) = V_+(z)V_-(z)$ when z is between C_+ and C_-.

Equating (3.10) with the expression $V(z) = V_+(z)V_-(z)$ we have

$$\frac{V_+(z)}{B(z)} = \frac{A(z)}{V_-(z)} z(z^2 - t^2). \tag{3.16}$$

The LHS (RHS) is an ANZ function of z inside C_+ (outside C_-), which is bounded as $|z| \to \infty$ and so the function must be a constant, c_1 say. Thus

$$V_+(z) = c_1 B(z) \tag{3.17}$$

$$V_-(z) = c_1^{-1} z(z^2 - t^2)A(z). \tag{3.18}$$

When $|z| < 1$, we proceed in the same way. Draw the curves C'_+ and C'_-, C'_+ inside the unit circle, C'_- inside C'_+, and with α and all the zeros of $A(z)$ inside C'_-.

In the limit $x \to 0$, $|1/r(z)| \sim z^N$, so $|1/r(z)| < 1$. Thus $\ln[1 + 1/r(z)]$ can be chosen to be single-valued and analytic between and on C'_+ and C'_-. We can then Wiener–Hopf factorize $1 + 1/r(z)$ by defining the functions $P'_+(z)$ and $P'_-(z)$ as

$$\ln P'_\pm(z) = \pm\frac{1}{2\pi \mathrm{i}} \oint_{C'_\pm} \ln\left[1 + \frac{1}{r(z')}\right] \frac{\mathrm{d}z'}{z' - z} \tag{3.19}$$

where $P'_+(z)$ is ANZ inside C'_+, $P'_-(z)$ is ANZ for z outside C'_-. As $|z| \to \infty$, $P'_-(z) \to 1$. When z is in the annulus between C'_+ and C'_-, Cauchy's integral formula now implies

$$1 + \frac{1}{r(z)} = P'_+(z)\, P'_-(z) = -\frac{V(z)\tilde{Q}(z)}{\tilde{Q}(-zx^{-1})(1 - z^2x)^N}. \tag{3.20}$$

Define $V'_+(z)$ and $V'_-(z)$ as follows:

$$V'_+(z) = P'_+(z)(1 - z^2x)^N/[\tilde{Q}_1(z)(z - \beta^{-1})] \tag{3.21}$$

$$V'_-(z) = P'_-(z)\tilde{Q}(-zx^{-1})/(z - \alpha). \tag{3.22}$$

We have now factorized $V(z)$ into two factors, $V'_+(z)$ which is is ANZ for z inside C'_+, and $V'_-(z)$ which is ANZ for z outside C'_-. When z is in the annulus between C'_+ and C'_-, we have the equality $V(z) = V'_+(z)V'_-(z)$.

When z is inside this annulus, we equate (3.10) with $V(z) = V'_+(z)V'_-(z)$ to get

$$\frac{V'_+(z)}{B(z)(z^2 - t^2)} = \frac{zA(z)}{V'_-(z)} \tag{3.23}$$

where now the LHS (RHS) is an ANZ function of z for z inside C'_+ (outside C'_-). Thus both sides of the equation are constant, c_2 say, and we have

$$V'_+(z) = c_2(z^2 - t^2)B(z) \tag{3.24}$$

$$V'_-(z) = c_2^{-1} zA(z). \tag{3.25}$$

From equations (3.17), (3.24) and (3.18), (3.25), we have the following:

$$V'_+(z) = (c_2/c_1)V_+(z)(z^2 - t^2) \tag{3.26}$$

$$V_-(z) = (c_1/c_2)V'_-(z)(z^2 - t^2)\,. \tag{3.27}$$

To evaluate the constant c_1/c_2, consider (3.27) in the limit $z \to \infty$; we noted earlier that $P_-(z)$, $P'_-(z) \to 1$ as $z \to \infty$, so from (3.5), (3.15) and (3.22) we deduce that

$$c_1/c_2 = 1\,. \tag{3.28}$$

We may use equations (3.26) and (3.27) to derive recurrence relations satisfied by $\tilde{Q}(z)$, which we can solve explicitly in the $N \to \infty$ limit.

From equations (3.14), (3.21) and (3.26), we deduce the recurrence relation

$$\tilde{Q}(z)\,\tilde{Q}(-zx) = (1 - z^2x)^N \frac{(z-\alpha)(z-\beta^{-1})}{(z^2-t^2)} \frac{P'_+(z)}{P_+(z)} \tag{3.29}$$

valid when z is inside C'_+. In the limit $N \to \infty$, the P_+ and P'_+ functions $\to 1$, so we find that $\tilde{Q}(z)$ is given by

$$\tilde{Q}(z) = \tilde{Q}(0) \prod_{m=1}^{\infty} \left(\frac{1 - z^2x^{4m-3}}{1 - z^2x^{4m-1}}\right)^N \frac{(1 - z^2t^{-2}x^{4m-2})}{(1 - z^2t^{-2}x^{4m-4})} \frac{(1 - z\alpha^{-1}x^{2m-2})}{(1 + z\alpha^{-1}x^{2m-1})} \frac{(1 - z\beta x^{2m-2})}{(1 + z\beta x^{2m-1})}\,. \tag{3.30}$$

This still contains the parameters t, α and β. From equation (3.29) in the $N \to \infty$ limit, setting $z = 0$ we note that

$$\left[\tilde{Q}(0)\right]^2 = -t^{-2}\alpha\beta^{-1}\,. \tag{3.31}$$

From equations (3.15), (3.22) and (3.27), we get the recurrence relation

$$\tilde{Q}(z)\,\tilde{Q}(-zx^{-1}) = (1 - z^2x^{-1})^N \frac{(z-\alpha)(z-\beta^{-1})}{(z^2-t^2)} \frac{P_-(z)}{P'_-(z)} \tag{3.32}$$

which is valid for z outside C_-. Taking the limit $N \to \infty$ once more, so that the functions $P_-(z)$ and $P'_-(z) \to 1$, we get

$$\tilde{Q}(z) = z^N \prod_{m=1}^{\infty} \left(\frac{1 - z^{-2}x^{4m-3}}{1 - z^{-2}x^{4m-1}}\right)^N \frac{(1 - z^{-2}t^2x^{4m-2})}{(1 - z^{-2}t^2x^{4m-4})} \frac{(1 - z^{-1}\alpha x^{2m-2})}{(1 + z^{-1}\alpha x^{2m-1})} \frac{(1 - z^{-1}\beta^{-1}x^{2m-2})}{(1 + z^{-1}\beta^{-1}x^{2m-1})}\,. \tag{3.33}$$

To derive an expression for $V(z)$ valid between C_+ and C'_-, using equation (3.27), we have

$$\begin{aligned} V(z) &= V_+(z)V'_-(z)(z^2 - t^2) \\ &= \tilde{Q}(-zx)\tilde{Q}(-zx^{-1})(z^2 - t^2)/[(z-\alpha)(z-\beta^{-1})]\,. \end{aligned} \tag{3.34}$$

We use (3.30) for $\tilde{Q}(-zx)$ and (3.33) for $\tilde{Q}(-zx^{-1})$, and substitute into (3.34). This produces a lengthy expression for $V(z)$ involving the parameters α, β and t, which simplifies when one considers the oddness of $V(z)$. The poles of $V(z)$ must occur in pairs, and this is only possible if α and β are related by

$$\alpha\beta = -x\,. \tag{3.35}$$

Substituting this in, the infinite products involving α and β cancel, and we get, from (3.6) and (3.34)

$$\Lambda(v) = G(z/t)\,(\rho/2x)^N \prod_{m=1}^{\infty} \left(\frac{1 - z^2x^{4m-1}}{1 - z^2x^{4m+1}} \frac{1 - z^{-2}x^{4m-1}}{1 - z^{-2}x^{4m+1}}\right)^N \tag{3.36}$$

where

$$G(z) = \pm \mathrm{i} x^{1/2}(z - z^{-1}) \prod_{m=1}^{\infty} \left(\frac{1 - z^2 x^{4m}}{1 - z^2 x^{4m-2}} \frac{1 - z^{-2} x^{4m}}{1 - z^{-2} x^{4m-2}} \right) . \tag{3.37}$$

This expression for the eigenvalue is still dependent on the parameter t, different values of t corresponding to different eigenvalues of the transfer matrix. All we know about t so far is that it is bounded as $x \to 0$, and that it lies on the unit circle in the $x \to 0$ limit. We shall now show that it in fact remains exactly on the unit circle as x increases.

We substitute into the functional relation (3.7), using (3.30) and (3.33) to get an expression for the product $\tilde{Q}(z)V(z)$ which is valid when z is in the annulus between C_+ and C'_-. Substituting into (3.7), the function on the right-hand side is equal to zero when z is one of the $N-2$ zeros of $\tilde{Q}_1(z)$, or when $z = \pm t$. For the latter case, substituting $z = t$ and $-t$, and dividing the resulting equations, we arrive at the following relation between α, x and t:

$$\alpha^2 = -t^2 x \tag{3.38}$$

which means that t must satisfy

$$[\phi(t)]^N = \pm 1 \tag{3.39}$$

where $\phi(t)$ is given by

$$\phi(t) = t \prod_{m=1}^{\infty} \left(\frac{1 - t^2 x^{4m-1}}{1 - t^2 x^{4m-3}} \frac{1 - t^{-2} x^{4m-3}}{1 - t^{-2} x^{4m-1}} \right) . \tag{3.40}$$

This implies that t lies on the unit circle for all x, there being $2N$ possible choices for t. The partition function depends on t only via t^2, so there are only N distinct eigenvalues. The right-hand side of (3.7) also vanishes when z is a zero of $\tilde{Q}_1(z)$ so in the same way we show that the zeros of $\tilde{Q}_1(z)$ lie exactly on the unit circle for all x. As the zeros lie exactly on the unit circle, we may shift the curves C_- and C'_+ so that they just surround the unit circle. Hence our expressions for $\tilde{Q}(z)$ are valid all the way up to the unit circle; equation (3.30) is valid for $|z| < 1$, and (3.33) is valid for $|z| > 1$.

We now evaluate the partition function, as defined in (3.1), in the large-lattice limit. When v is real, the eigenvalues (3.36) are complex, so as $N \to \infty$, the partition function, a sum over the N eigenvalues defined by (3.39), becomes an integral over all the allowed values of t,

$$Z = \oint \rho(t) \, [\Lambda(v)]^M \, \mathrm{d}t \tag{3.41}$$

where the integral is taken around the unit circle, and $\rho(t)$ is some distribution function, independent of N and M. Substituting (3.34) into (3.41) then gives an expression for Z. (The number of rows M is even to ensure periodic boundary conditions vertically, and so the $\pm$ sign in (3.37) is irrelevant.)

The eigenvalue (3.36) contains two distinct types of factors; those that are powers of N, and those that are not. The terms that increase exponentially with N contribute to the bulk part of the partition function, the free energy per site in the thermodynamic limit. This factor is also independent of t, and can be taken out of the integral (3.41). The integral is then independent of N, so we have, from (3.2)

$$\mathrm{e}^{-f/k_B T} = (\rho/2x) \prod_{m=1}^{\infty} \left(\frac{1 - z^2 x^{4m-1}}{1 - z^2 x^{4m+1}} \frac{1 - z^{-2} x^{4m-1}}{1 - z^{-2} x^{4m+1}} \right) \tag{3.42}$$

for the free energy per site in the thermodynamic limit. This result agrees with the result for periodic boundary conditions (equations (8.9.9) and (8.9.10) of [1]).

From equation (3.2), the other factors in (3.34) make up the interfacial tension, given by

$$e^{-Ms/k_BT} = \oint \rho(t)[G(z/t)]^M \,dt\,. \tag{3.43}$$

For M sufficiently large, we may evaluate this integral using saddle-point integration. The integral is given by the value of the integrand at its saddle point, together with some multiplicative factors that we can disregard as $M \to \infty$. The function G satisfies the relation

$$G(z) = G(-1/z) \tag{3.44}$$

which implies that the function has a saddle point when $z = \pm i$. Hence the integrand in (3.43) is maximized when

$$t = t_{\rm saddle} = \pm iz\,. \tag{3.45}$$

As z is arbitrary, the saddle points may lie off the unit circle; they will however lie inside the annulus between C_+ and C'_- because of the restriction (3.4), and so we will be able to deform the contour to pass through these points. Hence we arrive at the final result

$$e^{-s/k_BT} = 2x^{1/2}\prod_{m=1}^{\infty}\left(\frac{1+x^{4m}}{1+x^{4m-2}}\right)^2. \tag{3.46}$$

Acknowledgment

MTB and CMY thank the Australian Research Council for financial support.

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